

# Two-dimensional models

Contribution proposed for the project “Encyclopedia of Mathematical Physics”

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April 2005

## 1 History and motivation

Local quantum physics of systems with infinitely many interacting degrees of freedom leads to situations whose understanding often requires new physical intuition and mathematical concepts beyond that acquired in quantum mechanics and perturbative constructions in quantum field theory. In this situation two-dimensional soluble models turned out to play an important role. On the one hand they illustrate new concepts and sometimes remove misconceptions in an area where new physical intuition is still in process of being formed. On the other hand rigorously soluble models confirm that the underlying physical postulates are mathematically consistent, a task which for interacting systems with infinite degrees of freedom is mostly beyond the capability of pedestrian methods or brute force application of hard analysis on models whose natural invariances has been mutilated by a cut-off.

In order to underline these points and motivate the interest in 2-dimensional QFT, let us briefly look at the history, in particular at the physical significance of the three oldest two-dimensional models of relevance for statistical mechanics and relativistic particle physics, in chronological order: the Lenz-Ising model, Jordan’s model of bosonization/fermionization and the Schwinger model (QED<sub>2</sub>).

The Lenz-Ising (L-I) model was proposed in 1920 by Wilhelm Lenz [1] as the simplest discrete statistical mechanics model with a chance to go beyond the P. Weiss phenomenological Ansatz involving long range forces and instead explain ferromagnetism in terms of non-magnetic short range interactions. Its one-dimensional version was solved 4 years later by his student Ernst Ising. In his 1925 university of Hamburg thesis, Ising [1] not only showed that his chain solution could not account for ferromagnetism, but he also proposed some (as it turned out much later) not entirely correct intuitive arguments to the extend that this situation prevails to the higher dimensional lattice version. His advisor

Lenz as well as Pauli (at that time Lenz's assistant) accepted these reasonings and as a result there was considerable disappointment among the three which resulted in Ising's decision (despite Lenz's high praise of Ising's thesis) to look for a career outside of research. For many years a reference by Heisenberg [2] (to promote his own proposal as an improved description of ferromagnetism) to Ising's negative result was the only citation; the situation begun to change when Peierls [2] drew attention to "Ising's solution" and the results of Kramers and Wannier [2] cast doubts on Ising's intuitive arguments beyond the chain solution. The rest of this fascinating episode i.e. Lars Onsager's rigorous two-dimensional solution exhibiting ferromagnetic phase transition, Brucia Kaufman's simplification which led to conceptual and mathematical enrichments (as well as later contributions by many other illustrious personalities) hopefully remains a well-known part of mathematical physics history even beyond my own generation.

This work marks the beginning of applying rigorous mathematical physics methods to solvable two-dimensional models as the ultimate control of intuitive arguments in statistical mechanics and quantum field theory. The L-I model continued to play an important role in the shaping of ideas about universality classes of critical behavior; in the hands of Leo Kadanoff it became the key for the development of the concepts about order/disorder variables (The microscopic version of the famous Kramers-Wannier duality) and also of the operator product expansions which he proposed as a concrete counterpart to the more general field theoretic setting of Ken Wilson. Its massless version (and the related so-called Coulomb gas representation) became a role model in the setting of the BPZ minimal chiral models (Belavin, Polyakov and Zamolodchikov 1984) and it remained up to date the only model with non-abelian braid group (plektonic) statistics for which the  $n$ -point correlators can be written down explicitly in terms of elementary functions [2]. Chiral theories confirmed the pivotal role of "exotic" statistics [2] in low dimensional QFT by exposing the appearance of braid group statistics as a novel manifestation of Einstein causality [2].

Another conceptually rich model which lay dormant for almost two decades as the result of a misleading speculative higher dimensional generalization by its protagonist is the bosonization/fermionization model first proposed by Pascual Jordan [3]. This model establishes a certain equivalence between massless two-dimensional Fermions and Bosons; it is related to Thirring's massless 4-fermion coupling model and also to Luttinger's one-dimensional model of an electron gas [2]. One reason why even nowadays hardly anybody knows Jordan's contribution is certainly the ambitious but unfortunate title "the neutrino theory of light" under which he published a series of papers; besides some not entirely justified criticism of content, the reaction of his contemporaries consisted in a good-humored carnivalesque Spottlied (mockery song) about its title [2]. The massive version of the related Thirring model became the role model of integrable relativistic QFT and shed additional light on two-dimensional bosonization [6].

Both discoveries demonstrate the usefulness of having controllable low-dimensional models; at the same time their complicated history also illustrates the danger of rushing to premature "intuitive" conclusions about extensions to higher

dimensions. The search for the appropriate higher dimensional analog of a 2-dimensional observation is an extremely subtle endeavour. In the aforementioned two historical examples the true physical message of those models only became clear through hard mathematical work and profound conceptual analysis by other authors many years after the discovery of the original model.

A review of the early historical benchmarks of conceptual progress through the study of solvable two-dimensional models would be incomplete without mentioning Schwinger’s proposed solution [5] of two-dimensional quantum electrodynamics, afterwards referred to as the Schwinger model. Schwinger used this model in order to argue that gauge theories are not necessarily tied to zero mass vector particles. Some work was necessary [2] to unravel its physical content with the result that the would-be charge of that QED<sub>2</sub> model was “screened” and its apparent chiral symmetry broken; in other words the model exists only in the so-called Schwinger-Higgs phase with massive free scalar particles accounting for its physical content. Another closely related aspect of this model which also arose in the Lagrangian setting of 4-dimensional gauge theories was that of the  $\theta$ -angle parametrizing, an ambiguity in the quantization.

Thanks to its property of being superrenormalizable, the Schwinger model also served as a useful testing ground for the Euclidean integral formulation in the presence of Atiyah-Singer zero modes and their role in the Schwinger-Higgs chiral symmetry breaking [2]. These classical topological aspects of the functional integral formulation attracted a lot of attention beginning in the late 70s and through the 1980s but, as most geometrical aspects of the Euclidean functional integral representations, their intrinsic physical significance remained controversial<sup>1</sup>. This is no problem in the operator algebra approach where no topological or differential geometrical property is imposed but certain geometric structures (spacetime- and internal- symmetry properties) are encoded in the causality and spectral principles of observable algebras.

A coherent and systematic attempt at a mathematical control of two-dimensional models came in the wake of Wightman’s first rigorous programmatic formulation of QFT [2]. This formulation stayed close to the ideas underlying the impressive success of renormalized QED perturbation theory, although it avoided the direct use of Lagrangian quantization. The early attempts towards a “constructive QFT” found their successful realization in two-dimensional QFT (the  $P\varphi_2$  models [8]). Only in low dimensional theories the presence of Hilbert space positivity and energy positivity can be reconciled with the kind of mild short distance singularity behavior (superrenormalizability) which this functional analytic method requires. For this reason we will focus our main attention on alternative constructive methods which are free of this restrictions; they have the additional advantage to reveal more about the conceptual structure of QFTs beyond the mere assertion of their existence. The best illustration of the constructive power of these new methods comes from massless  $d=1+1$  conformal

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<sup>1</sup>Even in those superrenormalizable 2-dim models, where the measure theory underlying Euclidean functional integration can be mathematically controlled [8], there is no good reason why within this measure theoretical setting outside of quasiclassical approximations topological properties derived from continuity requirements should assert themselves.

and chiral QFT as well as from massive factorizing models. Their presentation and that of the conceptual message they contain for QFT in general will form the backbone of this article.

There are several books and review articles [9] on  $d=1+1$  conformal as well as on massive factorizing models [6][7]. To the extent that concepts and mathematical structures are used which permit no extension to higher dimensions (Kac-Moody algebras, loop groups, integrability, presence of an infinite number of conservation laws), this line of approach will not be followed in this report since our primary interest will be the use of two-dimensional models of QFT as “theoretical laboratories” of general QFT. Our aim is two-fold; on the one hand we intend to illustrate known principles of general QFT in a mathematically controllable context and on the other hand we want to identify new concepts whose adaptation to QFT in  $d=1+1$  lead to their solvability. In emphasizing the historical side of the problem, I also hope to uphold the awareness of the unity and historical continuity in QFT in times of rapidly changing fashions.

Although this article deals with problems of mathematical physics, the style of presentation is more on a narrative side as expected from an encyclopedia of mathematical physics contribution. I have tried to amend for the lack of references (which treat conformal and factorizing models under one roof as part of QFT) by referring at many instances to a broader review article [2] in which most of the left out references and additional related informations can be found. This review was especially written to serve as a reference which permits me to maintain an equilibrium between the present size of text and references.

## 2 General concepts and their two-dimensional manifestation

The general framework of QFT, to which the rich world of controllable two-dimensional models contributes as an important testing ground, exists in two quite different but nevertheless closely related formulations: the 1956 approach in terms of pointlike covariant fields due to Wightman [10], and the more algebraic setting which can be traced back to ideas which Haag developed shortly after [11] and which are based on spacetime-indexed operator algebras and related concepts which developed over a long period of time with contributions of many other authors into what is now referred to as algebraic QFT (AQFT) or simply local quantum physics (LQP). Whereas the Wightman approach aims directly at the (not necessarily observable) quantum fields, the operator algebraic setting ( $\rightarrow$  (78), *Algebraic approach to quantum field theory*) is more ambitious. It starts from physically well-motivated assumptions about the algebraic structure of local observables and aims at the reconstruction of the full field theory (including the operators carrying the superselected charges) in the spirit of a local representation theory of (the assumed structure of the) local observables. This has the advantage that the somewhat mysterious concept of an inner symmetry (as opposed to outer (spacetime) symmetry) can be traced back to its

physical roots which is the representation theoretical structure of the local observable algebra ( $\rightarrow$  (88), *Symmetries of lower spacetime dimensions*). In the standard Lagrangian quantization approach the inner symmetry is part of the input (multiplicity indices of field components on which subgroups of  $U(n)$  or  $O(n)$  act linearly) and hence it is not possible to problematize this fundamental question. When in low-dimensional spacetime dimensions the sharp separation (the Coleman-Mandula theorems) of inner versus outer symmetry becomes blurred as a result of the appearance of braid group statistics, the standard Lagrangian quantization setting of most of the textbooks is inappropriate and even the Wightman framework has to be extended. In that case the algebraic approach is the most appropriate.

The important physical principles which are shared between the Wightman approach (WA) [10] and the operator algebra (AQFT) setting [11] are the space-like locality or Einstein causality (in terms of pointlike fields or algebras localized in causally disjoint regions) and the existence of positive energy representations of the Poincaré group implementing covariance and the stability of matter.

The observable algebra consists of a family of (weakly closed) operator algebras  $\{\mathcal{A}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{K}}$  indexed by a family of convex causally closed spacetime regions  $\mathcal{O}$  (with  $\mathcal{O}'$  denoting the spacelike complement and  $\mathcal{A}'$  the von Neumann commutant) which act in one common Hilbert space. Certain properties cannot be naturally formulated in the pointlike field setting (vis. Haag duality<sup>2</sup> for convex regions  $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$ ), but apart from those properties the two formulations are quite close; in particular for two-dimensional theories there are convincing arguments that one can pass between the two without imposing additional technical requirements.

The two above requirements are often (depending on what kind of structural properties one wants to derive) complemented by additional impositions which, although not carrying the universal weight of principles nevertheless represent natural assumptions whose violation, even though not prohibited by the principles, would cause paradigmatic attention and warrants special explanations. Examples are “weak additivity”, “Haag duality” and “the split property”. Weak additivity i.e. the requirement  $\bigvee \mathcal{A}(\mathcal{O}_i) = \mathcal{A}(\mathcal{O})$  if  $\mathcal{O} = \bigcup \mathcal{O}_i$  expresses the “global from amalgamating the local” aspect which is inherent in the “action in the neighborhood” property of fields.

Haag duality is the statement that the commutant of observables not only contains the algebra of the causal complement (Einstein causality) but is even exhausted by it i.e.  $\mathcal{A}(\mathcal{O}') = \mathcal{A}(\mathcal{O})'$ ; it is deeply connected to the measurement process and its violation in the vacuum sector for convex causally complete regions signals spontaneous symmetry breaking in the associated charge-carrying field algebra [11]. It always can be enforced (assuming that the wedge-localized algebras fulfill (1) below) by symmetry-reducing extension called Haag-dualization. Its violation for multi-local region reveals the charge content of the

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<sup>2</sup>Haag duality holds for observable algebras in the vacuum sector in the sense that any violation can be explained in terms of a spontaneously broken symmetry; in local theories it always can be enforced by dualization and the resulting Haag dual algebra has a charge superselection structure associated with the unbroken subgroup.

model via charge-anticharge splitting in the neutral observable algebra [2].

The split property for regions  $\mathcal{O}_i$  separated by a finite spacelike distance  $\mathcal{A}(\mathcal{O}_1 \cup \mathcal{O}_2) \simeq \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)$  (Doplicher-Longo 1984) is a result of the adaptation of the “finiteness of phase space cell” property of QM to QFT (the so-called “nuclearity property”). Related to the Haag duality is the local version of the “time slice property” (the QFT counterpart of the classical causal dependency property) sometimes referred to as “strong Einstein causality”  $\mathcal{A}(\mathcal{O}'') = \mathcal{A}(\mathcal{O})''$  [2].

One of the most astonishing achievements of the algebraic approach is the DHR theory of superselection sectors (Doplicher, Haag and Roberts, 1971) i.e. the realization that the structure of charged (non-vacuum) representations (with the superposition principle being valid only within one representation) and the spacetime properties of the fields which are the carriers of these generalized charges, including their spacelike commutation relation which lead to the particle statistics and also to their internal symmetry properties, are already encoded in the structure of the Einstein causal observable algebra ( $\rightarrow$  (87) Symmetries in quantum field theory: algebraic aspects). The intuitive basis of this remarkable result (whose prerequisite is locality) is that one can generate charged sectors by spatially separating charges in the vacuum (neutral) sector and disposing of the unwanted charges at spatial infinity [11].

An important concept which especially in  $d=1+1$  has considerable constructive clout is “modular localization”. It is a consequence of the above algebraic setting if either the net of algebras have pointlike field generators, or if the one-particle masses are separated by spectral gaps so that the formalism of time dependent scattering can be applied [2]; in conformal theories this property holds automatically in all spacetime dimensions. It rests on the basic observation ( $\rightarrow$  (19) Tomita-Takesaki modular theory) that a *standard pair*  $(\mathcal{A}, \Omega)$  of a von Neumann operator algebra and a vector<sup>3</sup> gives rise to a Tomita operator  $S$  through its star-operation whose polar decomposition yield two modular objects, a 1-parametric subgroup  $\Delta^{it}$  of the unitary group of operators in Hilbert space whose  $\text{Ad}$ -action defines the modular automorphism of  $(\mathcal{A}, \Omega)$  whereas the angular part  $J$  is the modular conjugation which maps  $\mathcal{A}$  into its commutant  $\mathcal{A}'$

$$\begin{aligned} S\Omega &= A^*\Omega, \quad S = J\Delta^{\frac{1}{2}} \\ J_W &= U(j_W) = S_{scat}J_0, \quad \Delta_W^{it} = U(\Lambda_W(2\pi t)) \\ \sigma_W(t) &:= \text{Ad}\Delta_W^{it} \end{aligned} \tag{1}$$

The standardness assumption is always satisfied for any field theoretic pair  $(A(\mathcal{O}), \Omega)$  of a  $\mathcal{O}$ -localized algebra and the vacuum state (as long as  $\mathcal{O}$  has a non-trivial causal disjoint  $\mathcal{O}'$ ) but it is only for the wedge region  $W$  that the modular objects have a physical interpretation in terms of the global symmetry group of the vacuum as specified in the second line (1); the modular unitary represents

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<sup>3</sup>Standardness means that the operator algebra of the pair  $(\mathcal{A}, \Omega)$  act cyclic and separating on the vector  $\Omega$ .

the  $W$ -associated boost  $\Lambda_W(\chi)$  and the modular conjugation implements the TCP-like reflection along the edge of the wedge (Bisognano-Wichmann 1976). The third line is the definition of the modular group. Its usefulness results from the fact that it does not depend on the state vector  $\Omega$  but only on the state  $\omega(\cdot) = (\Omega, \cdot \Omega)$  which it induces, as well as the fact that the modular group  $\sigma^{(\eta)}(t)$  associated with a different state  $\eta(\cdot)$  on is unitarily equivalent to  $\sigma^{(\omega)}(t)$  with a unitary  $u(t)$  which fulfills the Connes cocycle property. The importance of this theory for local quantum physics results from the fact that it leads to concept of modular localization, a new intrinsic new scenario for field theoretic constructions which is different from the Lagrangian quantization schemes [2].

A special feature of  $d=1+1$  Minkowski spacetime is the disconnectedness of the right/left spacelike region leading to a right-left ordering structure. So in addition to the Lorentz invariant timelike ordering  $x \prec y$  ( $x$  earlier than  $y$ , which is independent of spacetime dimensions), there is an invariant spacelike ordering  $x < y$  ( $x$  to the left of  $y$ ) in  $d=1+1$  which opens the possibility of more general Lorentz-invariant spacelike commutation relation than those implemented by Bose/Fermi fields e.i. of fields with a spacelike braid group commutation structure. The appearance of such exotic statistics fields is not compatible with their Fourier transforms being creation/annihilation operators for Wigner particles; rather the state vectors which they generate from the vacuum contain in addition to the one-particle contribution a vacuum polarization cloud [2]. This close connection between new kinematic possibilities and interactions is one of the reasons why, different from higher dimensions where interactions are prescribed by the recipe of local couplings of free fields, low dimensional QFT offers a more intrinsic access to the central issue of interactions.

Although the operator-algebraic formulation is well-suited to such a more intrinsic approach, this does not mean that pointlike covariant fields have become less useful. They only changed their role; instead of mediating between classical and quantum field theory in the process of (canonical or functional integral) quantization, they now are universal generators of all local algebras and hence also of all modular objects  $\Delta_{\mathcal{O}}^{it}, J_{\mathcal{O}}$  which taken together form an infinite dimensional noncommutative unitary group in the Hilbert space. This universal group generated by the modular unitaries contains in particular the global spacetime symmetry group of the vacuum (Poincaré transformations, conformal transformations) as well as “partial diffeomorphisms” (section 8).

### 3 Boson/Fermion equivalence and superselection theory in a special model

The simplest and oldest but conceptually still rich model is obtained, as first proposed by Pascual Jordan [3], by using a 2-dim. massless Dirac current and showing that it may be expressed in terms of scalar canonical Bose cre-

ation/annihilation operators

$$j_\mu =: \bar{\psi} \gamma_\mu \psi := \partial_\mu \phi, \phi := \int_{-\infty}^{+\infty} \{e^{ipx} a^*(p) + h.c.\} \frac{dp}{2|p|} \quad (2)$$

Although the potential  $\phi(x)$  of the current as a result of its infrared divergence is not a field in the standard sense of an operator-valued distribution in the Fock space of the  $a(p)^{\#4}$ , the formal exponential defined as the zero mass limit of a well-defined exponential free massive field

$$: e^{i\alpha\phi(x)} := \lim_{m \rightarrow 0} m^{\frac{\alpha^2}{2}} : e^{i\alpha\phi_m(x)} : \quad (3)$$

turns out to be a bona fide quantum field in a larger Hilbert space (which extends the Fock space generated from applying currents to the vacuum). The power in front is determined by the requirement that all Wightman functions (computed with the help of free field Wick combinatorics) stay finite in this massless limit; the necessary and sufficient condition for this is the charge conservation rule

$$\left\langle \prod_i : e^{i\alpha_i \phi(x)} : \right\rangle = \begin{cases} \prod_{i < j} \left( \frac{-1}{(\xi_{+ij})_\varepsilon (\xi_{-ij})_\varepsilon} \right)^{\frac{1}{2} \alpha_i \alpha_j}, & \sum \alpha_i = 0 \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

where the resulting correlation function has been factored in terms of lightray coordinates  $\xi_{\pm ij} = x_{\pm i} - x_{\pm j}$ ,  $x_{\pm} = t \pm x$  and the  $\varepsilon$ -prescription stands for taking the standard Wightman  $t \rightarrow t + i\varepsilon$ ,  $\lim_{\varepsilon \rightarrow 0}$  boundary value which insures the positive energy condition. The additional presence in the vacuum expectation values of an arbitrary polynomial in the current  $\prod_i j_{\mu i}(y_i)$  does not change the argument leading to the charge conservation law 4. The finiteness of the limit insures that the resulting zero mass limiting theory is a bona fide quantum field theory i.e. its system of Wightman functions which permits the construction of an operator theory in a Hilbert space with a distinguished vacuum vector. There exists another very intuitive and physically more intrinsic method in which one stays in the zero mass setting and obtains the charged sectors by splitting neutral operators as  $expij(f)$  belonging to the vacuum sector and “dumping the unwanted compensating charge behind the moon” [11] by taking suitable sequences of test function and adjusting normalizations appropriately.

The factorization into lightray components (4) shows that the exponential charge-carrying operators inherit this factorization into two independent chiral components  $:expi\alpha\phi(x) =:expi\alpha\phi_+(x_+)::expi\alpha\phi_-(x_-):$  each one being invariant under scaling  $\xi \rightarrow \lambda\xi$  if one assigns the scaling dimension  $d = \frac{\alpha^2}{2}$  to the chiral exponential field and  $d = 1$  to the current. As any Wightman field this is a singular object which only after smearing with Schwartz test functions yields an (unbounded) operator. But the above form of the correlation function belongs to a class of distributions which admits a much larger test function space consisting of smooth functions which instead of decreasing rapidly only need to

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<sup>4</sup>It becomes an operator after smearing with test functions whose Fourier transform vanishes at  $p=0$ .



be bounded so that they stay finite on the compactified lightray line  $\dot{R} = S^1$ . To make this visible one uses the Cayley transform (now  $x$  denotes either  $x_+$  or  $x_-$ )

$$z = \frac{1 + ix}{1 - ix} \in S^1 \quad (5)$$

This transforms the Schwartz test function into a space of test functions on  $S^1$  which have an infinite order zero at  $z = -1$  (corresponding to  $x = \pm\infty$ ) but the rotational transformed fields  $j(z), :exp i\alpha\phi(z):$  permit the smearing with all smooth functions on  $S^1$ , a characteristic feature of all conformal invariant theories as the present one turns out to be. There is an additional advantage in the use of this compactification. Fourier transforming the circular current actually allows for a quantum mechanical zero mode whose possible non zero eigenvalues indicate the presence of additional charge sectors beyond the charge zero vacuum sector. For the exponential field this leads to a quantum mechanical pre-exponential factor which *automatically* insures the charge selection rules (in agreement with the non availability of the “compensating charge behind the moon” argument) so that unrestricted (by charge conservation) Wick contraction rules can be applied. In this approach the original chiral Dirac Fermion  $\psi(x)$  (from which the current was formed as the  $:\bar{\psi}\psi:$  composite) re-appears as a charge-carrying exponential field for  $\alpha = 1$  and thus illustrates the meaning of bosonization/fermionization<sup>5</sup>. Naturally this terminology has to be taken with a grain of salt in view of the fact that the bosonic current algebra only generates a superselected subspace into which the charge-carrying exponential field does not fit. Only in the case of massive 2-dim. QFT Fermions can be incorporated into a Fock space of Bosons (see last section). At this point it should however be clear to the reader that the physical content of Jordan’s paper had nothing to do with its misleading title “neutrino theory of light” but rather was a special illustration about charge superselection rules in QFT, long before this general concept was recognized and formalized.

A systematic and rigorous approach consists in solving the problem of positive energy representation theory for the Weyl algebra<sup>6</sup> on the circle (which is the rigorous operator algebraic formulation of the abelian current algebra). It is the operator algebra generated by the exponential of a smeared chiral current (always with real test functions) with the following relation between the generators

$$\begin{aligned} W(f) &= e^{ij(f)}, \quad j(f) = \int \frac{dz}{2\pi i} j(z) f(z), \quad [j(z), j(z')] = -\delta'(z - z'), \quad (6) \\ W(f)W(g) &= e^{-\frac{1}{2}s(f,g)} W(f+g), \quad W^*(f) = W(-f) \\ \mathcal{A}(S^1) &= \text{alg} \{W(f), f \in C_\infty(S^1)\}, \quad \mathcal{A}(I) = \text{alg} \{W(f), \text{supp} f \subset I\} \end{aligned}$$

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<sup>5</sup>It is interesting to note that Jordan’s original treatment [3] of fermionization had such a pre-exponential quantum mechanical factor.

<sup>6</sup>The Weyl algebra originated in quantum mechanics around 1927; its use in QFT only appeared after the cited Jordan paper. By representation we mean here a regular representation in which the exponentials can be differentiated in order to obtain (unbounded) smeared current operators.

where  $s(.,.) = \int \frac{dz}{2\pi i} f'(z)g(z)$  is the symplectic form which characterizes the Weyl algebra structure and the last line denotes the unique  $C^*$  algebra generated by the unitary objects  $W(f)$ . A particular representation of this algebra is given by assigning the vacuum state to the generators  $\langle W(f) \rangle_0 = e^{-\frac{1}{2}\|f\|_0^2}$ ,  $\|f\|_0^2 = \sum_{n \geq 1} n |f_n|^2$ . Starting with the vacuum Hilbert space representation  $\mathcal{A}(S^1)_0 = \pi_0(\mathcal{A}(S^1))$  one easily checks that the formula

$$\begin{aligned} \langle W(f) \rangle_\alpha &:= e^{i\alpha f_0} \langle W(f) \rangle_0 \\ \pi_\alpha(W(f)) &= e^{i\alpha f_0} \pi_0(W(f)) \end{aligned} \quad (7)$$

defines a state with positive energy i.e. one whose GNS representation for  $\alpha \neq 0$  is unitarily inequivalent to the vacuum representation. Its incorporation into the vacuum Hilbert space (second line) is part of the DHR formalism. It is convenient to view this change as the result of an application of an automorphism  $\gamma_\alpha$  on the  $C^*$ -Weyl algebra  $\mathcal{A}(S^1)$  which is implemented by a unitary charge generating operator  $\Gamma_\alpha$  in a larger (nonseparable) Hilbert space which contains all charge sectors  $H_\alpha = \Gamma_\alpha H_0$ ,  $H_0 \equiv H_{vac} = \overline{\mathcal{A}(S^1)\Omega}$

$$\langle W(f) \rangle_\alpha = \langle \gamma_\alpha(W(f)) \rangle_0, \quad \gamma_\alpha(W(f)) = \Gamma_\alpha W(f) \Gamma_\alpha^* \quad (8)$$

$\Gamma_\alpha \Omega = \Omega_\alpha$  describes a state with a rotational homogeneous charge distribution; arbitrary charge distributions  $\rho_\alpha$  of total charge  $\alpha$  i.e.  $\int \frac{dz}{2\pi i} \rho_\alpha = \alpha$  are obtained in the form

$$\psi_{\rho_\alpha}^\zeta = \eta(\rho_\alpha) W(\hat{\rho}_\alpha^\zeta) \Gamma_\alpha \quad (9)$$

where  $\eta(\rho_\alpha)$  is a numerical phase factor and the net effect of the Weyl operator is to change the rotational homogeneous charge distribution into  $\rho_\alpha$ . The necessary charge-neutral compensating function  $\hat{\rho}_\alpha^\zeta$  in the Weyl cocycle  $W(\hat{\rho}_\alpha^\zeta)$  is uniquely determined in terms of  $\rho_\alpha$  up to the choice of one point  $\zeta \in S^1$  (the determining equation involves the  $\ln z$  function which needs the specification of a branch cut [2]). From this formula one derives the commutation relations  $\psi_{\rho_\alpha}^\zeta \psi_{\rho_\beta}^\zeta = e^{\pm i\pi\alpha\beta} \psi_{\rho_\beta}^\zeta \psi_{\rho_\alpha}^\zeta$  for spacelike separations of the  $\rho$  supports; hence these fields are relatively local (bosonic) for  $\alpha\beta = 2\mathbb{Z}$ . In particular if only one type of charge is present, the generating charge is  $\alpha_{gen} = \sqrt{2N}$  and the composite charges are multiples i.e.  $\alpha_{gen}\mathbb{Z}$ . This locality condition providing bosonic commutation relations does not yet insure the  $\zeta$ -independence. Since the equation which controls the  $\zeta$ -change turns out to be

$$\psi_{\rho_\alpha}^{\zeta_1} (\psi_{\rho_\alpha}^{\zeta_2})^* = e^{\pm i\pi\alpha\beta} e^{2\pi i Q \alpha} \quad (10)$$

one achieves  $\zeta$ -independence by restricting the Hilbert space charges to be “dual” to that of the operators i.e.  $Q = \left\{ \frac{1}{\sqrt{2N}} \mathbb{Z} \right\}$ . The localized  $\psi_{\rho_\alpha}^{\zeta_1}$  operators acting on the restricted separable Hilbert space  $H_{res}$  generate a  $\zeta$ -independent extended observable algebra  $\mathcal{A}_N(S^1)$  [2] and it is not difficult to see that its representation in  $H_{res}$  is reducible and that it decomposes into  $2N$  charge sectors  $\left\{ \frac{1}{\sqrt{2N}} n, n = 0, 1, \dots, N-1 \right\}$ . Hence the process of extension has

led to a charge quantization with a finite (“rational”) number of charges relative to the new observable algebra which is neutral in the new charge counting  $\frac{1}{\alpha_{gen}}\mathbb{Z}/\alpha_{gen}\mathbb{Z} = \mathbb{Z}/\alpha_{gen}^2 = \mathbb{Z}_{2N}$ . The charge-carrying fields in the new setting are also of the above form (9), but now the generating field carries the charge  $\int \frac{dz}{2\pi i} \rho_{gen} = Q_{gen}$  which is a  $\frac{1}{2N}$  fraction of the old  $\alpha_{gen}$ . Their commutation relations for disjoint charge supports are “braidal” (or better “plektonic”<sup>7</sup> which is more on par with bosonic/fermionic). These objects considered as operators localized on  $S^1$  do depend on the cut  $\zeta$ , but using an appropriate finite covering of  $S^1$  this dependence is removed [2]. So the field algebra  $\mathcal{F}_{\mathbb{Z}_{2N}}$  generated by the charge carrying fields (as opposed to the bosonic observable algebra  $\mathcal{A}_N$ ) has its unique localization structure on a finite covering of  $S^1$ . An equivalent description which gets rid of  $\zeta$  consists in dealing with operator-valued sections on  $S^1$ . The extension  $\mathcal{A} \rightarrow \mathcal{A}_N$ , which renders the Hilbert space separable and quantizes the charges, seems to be characteristic for abelian current algebra, in all other models which have been constructed up to now the number of sectors is at least denumerable and in the more interesting ones even finite (rational models). An extension is called maximal if there exists no further extension which maintains the bosonic commutation relation. For the case at hand this would require the presence of another generating field of the same kind as above which belongs to an integer  $N'$  is relatively local to the first one. This is only possible if  $N$  is divisible by a square.

In passing it is interesting to mention a somewhat unexpected relation between the Schwinger model, whose charges are *screened*, and the Jordan model. Since the Lagrangian formulation of the Schwinger model is a gauge theory, the analog of the 4-dim. *asymptotic freedom* wisdom would suggest the possibility of *charge liberation* in the short distance limit of this model. This seems to contradict the statement that the intrinsic content of the Schwinger model (QED<sub>2</sub> with massless Fermions) (after removing a classical degree of freedom<sup>8</sup>) is the QFT of a free massive Bose field and such a simple free field is at first sight not expected to contain subtle informations about asymptotic charge liberation. Well, as we have seen above, the massless limit really does have liberated charges and the short distance limit of the massive free field is the massless model [2].

As a result of the peculiar bosonization/fermionization aspect of the zero mass limit of the derivative of the massive free field, Jordan’s model is also closely related to the massless Thirring model (and the related Luttinger model for an interacting one-dimensional electron gas) whose massive version is in the class of factorizing models (see later section)<sup>9</sup>. The Thirring model is a special case in a vast class of “generalized” multi-coupling multi-component Thirring

<sup>7</sup>In the abelian case like the present the terminology “anyonic” enjoys widespread popularity: but in the present context the “any” does not go well with charge quantization.

<sup>8</sup>In its original gauge theoretical form the Schwinger model has an infinite vacuum degeneracy. The removal of this degeneracy (restoration of the cluster property) with the help of the “ $\theta$ -angle formalism” leaves a massive free Bose field (the Schwinger-Higgs mechanism). As expected in  $d=1+1$  the model only possesses this phase.

<sup>9</sup>Another structural consequence of this aspect leads to Coleman’s theorem [2] which connects the Mermin-Wagner no-go theorem for two-dimensional spontaneous continuous symmetry-breaking with these zero mass peculiarities.

models i.e. models with 4-Fermion interactions. Under this name they were studied in the early 70s [2] with the aim to identify massless subtheories for which the currents form chiral current algebras.

The counterpart of the potential of the conserved Dirac current in the massive Thirring model is the Sine-Gordon field, i.e. a composite field which in the attractive regime of the Thirring coupling again obeys the so-called Sine-Gordon equation of motion. Coleman gave a supportive argument [2] but some fine points about the range of its validity in terms of the coupling strength remained open<sup>10</sup>. A rigorous confirmation of these facts was recently given in the bootstrap-formfactor setting [2]. Massive models which have a continuous or discrete internal symmetry have “disorder” fields which implement a “half-space” symmetry on the charge-carrying field (acting as the identity in the other half axis) and together with the basic pointlike field form composites with have exotic commutation relations (see last subsection).

## 4 The conformal setting, structural results

Chiral theories play a special role within the setting of conformal quantum fields. General conformal theories have observable algebras which live on compactified Minkowski space ( $S^1$  in the case of chiral models) and fulfill the Huygens principle, which in an even number of spacetime dimension means that the commutator is only nonvanishing for lightlike separation of the fields. The fact that this classical wisdom breaks down for non-observable conformal fields (e.g. the massless Thirring field) was noticed at the beginning of the 70s and considered paradoxical at that time (“reverberation” in the timelike (Huygens) region). Its resolution around 1974/75 confirmed that such fields are genuine conformal covariant objects but that some fine points about their causality needed to be addressed. The upshot was the proposal of two different but basically equivalent concepts about *globally* causal fields. They are connected by the following global decomposition formula

$$A(x_{cov}) = \sum A_{\alpha,\beta}(x), \quad A_{\alpha,\beta}(x) = P_\alpha A(x) P_\beta, \quad Z = \sum e^{id_\alpha} P_\alpha \quad (11)$$

On the left hand side the spacetime point of the field is a point on the universal covering of the conformal compactified Minkowski space. These are fields (Luescher and Mack, 1976) [2] which “live” in the sense of quantum (modular) localization on the universal covering spacetime (or on a finite covering, depending on the “rationality” of the model) and fulfill the global causality condition previously discovered by I. Segal [2]. They are generally highly reducible with respect to the center of the covering group. The family of fields on the right hand side on the other hand are fields which were introduced (Schroer and Swieca, 1974) with the aim to have objects which live on the projection  $x(x_{cov})$  i.e. on the spacetime of the physics laboratory instead of the “hells and

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<sup>10</sup>It was noticed that the current potential of the free massive Dirac Fermion ( $g=0$ ) does not obey the Sine-Gordon equation [2].

heavens” of the covering [2]. They are operator-distributional valued *sections* in the compactification of ordinary Minkowski spacetime. The connection is given by the above decomposition formula into irreducible conformal blocks with respect to the center  $\mathbf{Z}$  of the noncompact covering group  $\widetilde{SO(2, n)}$  where  $\alpha, \beta$  are labels for the eigenspaces of the generating unitary  $Z$  of the abelian center  $\mathbf{Z}$ . The decomposition (11) is minimal in the sense that in general there generally will be a refinement due to the presence of additional charge superselection rules (and internal group symmetries). The component fields are not Wightman fields since they annihilate the vacuum if the right hand projection differs from  $P_0 = P_{vac}$ .

Note that the Huygens (timelike) region in Minkowski spacetime has an timelike ordering structure  $x \prec y$  or  $x \succ y$  (earlier, later). In  $d=1+1$  the topology allows in addition a spacelike left-right ordering  $x \leqslant y$ . In fact it is precisely the presence of this two orderings in conjunction with the factorization of the vacuum symmetry group  $\widetilde{SO(2, 2)} \simeq \widetilde{PSL(2R)_l} \otimes \widetilde{PSL(2, R)_r}$  in particular  $\mathbf{Z} = \mathbf{Z}_l \otimes \mathbf{Z}_r$ , which is at the root of a significant simplification. This situation suggested a tensor factorization into chiral components and led to an extremely rich and successful construction program of two-dimensional conformal QFT as a two-step process: the classification of chiral observable algebras on the lightray and the amalgamation of left-right chiral theories to 2-dimensional local conformal QFT. The action on the circular coordinates  $z$  is through fractional  $SU(1, 1)$  transformations  $g(z) = \frac{\alpha z + \beta}{\beta z + \bar{\alpha}}$  whereas the covering group acts on the Mack-Luescher covering coordinates.

The presence of an ordering structure permits the appearance of more general commutation relations for the above  $A_{\alpha\beta}$  component fields namely

$$A_{\alpha,\beta}(x)B_{\beta,\gamma}(y) = \sum_{\beta'} R_{\beta,\beta'}^{\alpha,\gamma} B_{\alpha,\beta'}(y)A_{\beta',\gamma}(x), \quad x > y \quad (12)$$

with numerical  $R$ -coefficients which, as a result of associativity and relative commutativity with respect to observable fields have to obey certain structure relations; in this way Artin braid relations emerge as a new manifestation of the Einstein causality principle for observables in low-dimensional QFT (Rehren and Schroer, 1989) [2]. Indeed the DHR method to interpret charged fields as charge-superselection carriers (tied by local representation theory to the bosonic local structure of observable algebras) leads precisely to such a *plektonic* statistics structure (Fredenhagen, Rehren and Schroer, 1992, Froehlich and Gabbiani 1993) for systems in low spacetime dimension ( $\rightarrow$ (88) Symmetries of lower spacetime dimensions). With an appropriately formulated adjustment to observables fulfilling the Huygens commutativity, this plektonic structure (but now disconnected from particle/field statistics) is also a possible manifestation of causality for the higher dimensional timelike structure [2].

Although the above presentation may have created the impression that there is a straight line from the decomposition theory of the early 70s to the construction of interesting models, the is not quite the way history unfolded. The only examples known up to the appearance of the seminal BPZ work (Belavin,

Polyakov and Zamolodchikov, 1984) were the abelian current models of the previous section which furnish a rather poor man's illustration of the richness of the decomposition theory. The floodgates of conformal QFT were only opened after the BPZ discovery of "minimal models" which was preceded by the observation (Friedan, Qiu and Shenker 1984) that the algebra of the stress-energy tensor came with a new representation structure which was not compatible with an underlying internal group symmetry ( $\rightarrow$ (87) Symmetries and conservation laws).

The importance of the stress-energy tensor in two-dimensional QFT in its role as a generator of a new infinite-dimensional Lie algebra was already recognized soon after Virasoro's extraction of part of this algebra from Veneziano's dual model, but the first field theoretic derivations were limited to the stress-energy tensor of a free massless Dirac field. There was however another more traditional line of structural arguments which originated in Wightman's formulation of QFT ( $\rightarrow$  (317) Axiomatic quantum field theory) wherein one was trying to go beyond free fields by staying close to free field algebraic structures. The first attempt beyond the generalized free field commutation relations was O. W. Greenberg's proposal to investigate fields with a Lie-type of spacetime commutation relations i.e. a set of fields  $A_i(x)$  fulfilling the "Lie relation" (Greenberg, 1961)

$$[A_i(x), A_j(y)] = c - \text{number} + \int C_{ij}^k(x, y, z) A_k(z) d^n z \quad (13)$$

The non-abelian chiral current algebras at the beginning of the 70s gave some obvious illustrations of this structure, but the more interesting case was that of the generic chiral stress-energy tensor [2]. Later it was shown (Baumann 1976) that there can be no (scalar) Lie fields in higher spacetime dimensions [2] i.e.  $C_{ij}^k(x, y, z) \equiv 0$ . Examples of conformal Lie fields are the current algebras and some of the so-called W-algebras (generalizations of the stress-energy algebra). We will see in section 7 that solvable (factorizable) massive two-dimensional theories are characterized by a different algebraic structure.

## 5 Chiral fields and 2-dimensional conformal models

Let us start with a family which generalizes the abelian model of the previous section. Instead of a one-component abelian current we now take  $n$  independent copies. The resulting multi-component Weyl algebra has the previous form except that the current is  $n$ -component and the real function space underlying the Weyl algebra consists of functions with values in an  $n$ -component real vector space  $f \in LV$  with the standard Euclidean inner product denoted by  $(,)$ . The local extension now leads to  $(\alpha, \beta) \in 2\mathbb{Z}$  i.e. an even integer lattice  $\mathcal{L}$  in  $V$ , whereas the restricted Hilbert subspace  $H_{\mathcal{L}^*}$  which ensures  $\zeta$ -independence is associated with the dual lattice  $L^* : (\lambda_i, \alpha_k) = \delta_{ik}$  which contains  $\mathcal{L}$ . The resulting superselection structure (i.e. the  $Q$ -spectrum) corresponds to the finite

factor group  $\mathcal{L}^*/\mathcal{L}$ . For selfdual lattices  $\mathcal{L}^* = \mathcal{L}$  (which only can occur if  $\dim V$  is a multiple of 8) the resulting observable algebra has only the vacuum sector; the most famous case is the Leech lattice  $\Lambda_{24}$  in  $\dim V = 24$ , also called the “moonshine” model. The observation that the root lattices of the Lie algebras of type  $A, B$  or  $E$  (example.  $su(n)$  corresponding to  $A_{n-1}$ ) also appear among the even integral lattices suggests that the nonabelian current algebras associated to those Lie algebras can also be implemented. This turns out to be indeed true as far as the level 1 representations are concerned which brings us to the second family: the nonabelian current algebras of level  $k$  associated to those Lie algebras; they are characterized by the commutation relation

$$[J_\alpha(z), J_\beta(z')] = if_{\alpha\beta}^\gamma j_\gamma(z) \delta(z - z') - \frac{1}{2} k g_{\alpha\beta} \delta'(z - z') \quad (14)$$

where  $f_{\alpha\beta}^\gamma$  are the structure constants of the underlying Lie algebra,  $g$  their Cartan-Killing form and  $k$ , the level of the algebra, must be an integer in order that the current algebra can be globalized to a loop group algebra. The Fourier decomposition of the current leads to the so called affine Lie algebras, a special family of Kac-Moody algebras. For  $k=1$  these currents can be constructed as bilinears in terms of multi-component chiral Dirac field; there exists also the mentioned possibility to obtain them by constructing their maximal Cartan currents within the above abelian setting and representing the remaining non-diagonal currents as certain charge-carrying (“vertex” algebra) operators. Level  $k$  algebras can be constructed from reducing tensor products of  $k$  level one currents or directly via the representation theory of infinite-dimensional affine Lie algebras<sup>11</sup>. Either way one finds that e.g. the  $SU(2)$  current algebra of level  $k$  has (together with the vacuum sector)  $k+1$  sectors (inequivalent representations). The different sectors are already distinguished by the structure of their ground states of the conformal Hamiltonian  $L_0$ . Although the computation of higher point correlation functions for  $k > 1$ , there is no problem in securing the existence of the algebraic nets which define these chiral models as well as their  $k+1$  representation sectors and to identify their generating charge-carrying fields (primary fields) including their R-matrices appearing in their plektonic commutation relations. It is customary to use the notation  $SU(2)_k$  for the abstract operator algebras associated with the current generators (14) and we will denote their  $k+1$  equivalence classes of representations by  $\mathcal{A}_{SU(2)_k, n}$ ,  $n = 0, \dots, k$ , whereas representations of current algebras for higher rank groups require a more complicated labeling (in terms of Weyl chambers).

The third family of models are the so-called minimal models which are associated with the Lie-field commutation structure of the chiral stress-energy tensor which results from the chiral decomposition of a conformally covariant 2-dimensional stress-energy tensor

$$[T(z), T(z')] = i(T(z) + T(z'))\delta'(z - z') + \frac{ic}{24\pi}\delta'''(z - z') \quad (15)$$

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<sup>11</sup>The global exponentiated algebras (the analogs to the Weyl algebra) are called loop group algebras.

whose Fourier decomposition yields the Witt-Virasoro algebra i.e. a central extension<sup>12</sup> of the Lie algebra of the  $Diff(S^1)$ . The first two coefficients are determined by the physical role of  $T(z)$  as the generating field density for the Lie algebra of the Poincaré group whereas the central extension parameter  $c > 0$  (positivity of the two-point function) for the connection with the generation of the Moebius transformations and the undetermined parameter  $c > 0$  (the central extension parameter) is easily identified with the strength of the two-point function. Although the structure of the T-correlation functions resembles that of free fields (in the sense that is a algebraically computable unique set of correlation functions once one has specified the two-point function), the realization that  $c$  is subject to a discrete quantization if  $c < 1$  came as a surprise. As already mentioned, the observation that the superselection sectors (the positive energy representation structure) of this algebra did not at all follow the logic of a representation theory of an inner symmetry group generated a lot of attention and stimulated a flurry of publications on symmetry concepts beyond groups (quantum groups). A concept of fundamental importance is the DHR theory of localized endomorphisms of operator algebras and the concept of operator algebraic inclusions in particular inclusions with conditional expectations (V. Jones inclusions)

The  $SU(2)_k$  current coset construction (Goddard, Kent and Olive 1986) revealed that the proof of existence and the actual construction of the minimal models is related to that of the  $SU(2)_k$  current algebras. Constructing chiral models does not necessarily mean the explicit determination of its n-point Wightman functions of their generating fields (which for most chiral models remains a prohibitively complicated) but rather a proof of their existence by demonstrating that these models are obtained from free fields by a series of computational complicated but mathematically controlled operator-algebraic steps as: reduction of tensor products, formation of orbifolds under group actions, coset constructions and a special kind of extensions. The generating fields of the models are nontrivial in the sense of not obeying free field equations (i.e. not being “on-shell”). The cases where one can write down explicit n-point functions of generating fields are very rare; in the case of the minimal family this is only possible for the Ising model [2].

To show the power of inclusion theory for the determination of the charge content of theory let us look at a simple illustration in the context of the above multi-component abelian current algebra. The vacuum representation of the corresponding Weyl algebra is generated from smooth  $V$ -valued functions on the circle modulo constant functions (i.e. functions with vanishing total integral)  $f \in LV_0$ . These functions equipped with the aforementioned complex structure and scalar product yield a Hilbert space. The  $I$ -localized subalgebra is generated by the Weyl image of  $I$ -supported functions (class functions whose representing

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<sup>12</sup>The presence of the central term in the context of QFT (the analog of the Schwinger term) was noticed later, however the terminology Witt-Virasoro algebra in the physics literature came to mean the Lie algebra of diffeomorphisms of the circle including the central extension.



functions are constant in the complement  $I'$ )

$$\begin{aligned}\mathcal{A}(I) &:= \text{alg} \{W(f) \mid f \in K(I)\} \\ K(I) &= \{f \in LV_0 \mid f = \text{const in } I'\}\end{aligned}\tag{16}$$

The one-interval Haag duality  $\mathcal{A}(I)' = \mathcal{A}(I')$  (the commutant algebra equals the algebra localized in the complement) is simply a consequence of the fact that the symplectic complement  $K(I)'$  in terms of  $\text{Im}(f, g)$  consists of real functions in that space which are localized in the complement i.e.  $K(I)' = K(I')$ . The answer to the same question for a double interval  $I = I_1 \cup I_3$  (think of the first and third quadrant on the circle) does not lead to duality but rather to a genuine inclusion

$$\begin{aligned}K((I_1 \cup I_3)') &= K(I_2 \cup I_4) \subset K(I_1 \cup I_3)' \\ &\curvearrowright K(I_1 \cup I_3) \subset K((I_1 \cup I_3)')'\end{aligned}\tag{17}$$

The meaning of the left hand side is clear, these are functions which are constant in  $I_1 \cup I_3$  with the same constant in the two intervals whereas the functions on the right hand side are less restrictive in that the constants can be different. The conversion of real subspaces into von Neumann algebras by the Weyl functor leads to the algebraic inclusion  $\mathcal{A}(I_1 \cup I_3) \subset \mathcal{A}((I_1 \cup I_3)')'$ . In physical terms the enlargement results from the fact that within the charge neutral vacuum algebra a charge split with one charge in  $I_1$  and the compensating charge in  $I_2$  for all values of the (unquantized) charge occurs. A more realistic picture is obtained if one allows a charge split is subjected to a charge quantization implemented by a lattice condition  $f(I_2) - f(I_4) \in 2\pi L$  which relates the two multi-component constant functions (where  $f(I)$  denotes the constant value  $f$  takes in  $I$ ). As in the previous one-component case the choice of even lattices corresponds to the local (bosonic) extensions. Although imposing such a lattice structure destroys the linearity of the  $K$ , the functions still define Weyl operators which generated operator algebras  $\mathcal{A}_L(I_1 \cup I_2)$ <sup>13</sup>. But now the inclusion involves the dual lattice  $L^*$  (which of course contains the original lattice)

$$\begin{aligned}\mathcal{A}_L(I_1 \cup I_2) &\subset \mathcal{A}_{L^*}(I_1 \cup I_2) \\ \text{ind} \{ \mathcal{A}_L(I_1 \cup I_2) \subset \mathcal{A}_L((I_1 \cup I_2)')' \} &= |G| \\ \mathcal{A}_L(I_1 \cup I_2) &= \text{inv}_G \mathcal{A}_{L^*}(I_1 \cup I_2)\end{aligned}$$

This time the possible charge splits correspond to the factor group  $G = L^*/L$  i.e. the number of possibilities is  $|G|$  which measures the relative size of the bigger algebra in terms of the smaller. This is a special case of the general concept of the so-called Jones index of an inclusion which is a numerical measure of its depth. A prerequisite is that the inclusion permits a conditional expectation which is a generalization of the averaging under the “gauge group”  $G$  on  $\mathcal{A}_{L^*}(I_1 \cup I_2)$  in the third line which identifies the invariant smaller algebra

<sup>13</sup>The linearity structure is recovered on the level of the operator algebra.

is the fix point algebra (the invariant part) under the action of  $G$ . In fact using the conceptual framework of Jones one can show that the two-interval inclusion is independent of the position of the disjoint intervals characterized by the group  $G$ .

There exists another form of this inclusion which is more suitable for generalizations. One starts from the quantized charge extended local algebra  $\mathcal{A}_L^{ext} \supset \mathcal{A}$  described before in terms of an integer even lattice  $L$  (which lives in the separable Hilbert space  $H_{L^*}$ ) as our observable algebra. Again the Haag duality is violated and converted into an inclusion  $\mathcal{A}_L^{ext}(I_1 \cup I_2) \subset \mathcal{A}_L^{ext}((I_1 \cup I_2)')'$  which turns out to have the same  $G = L^*/L$  charge structure (it is in fact isomorphic to the previous inclusion). In the general setting (current algebras, minimal model algebras,...) this double interval inclusion is particularly interesting if the associated Jones index is finite. One finds (Kawahigashi-Longo-Mueger 2001) [2]

**Theorem 1** *A chiral theory with finite Jones index  $\mu = \text{ind} \{ \mathcal{A}((I_1 \cup I_2)')' : \mathcal{A}(I_1 \cup I_2) \}$  for the double interval inclusion (always assuming that  $A(S^1)$  is strongly additive and split) is a rational theory and the statistical dimensions  $d_\rho$  of its charge sectors are related to  $\mu$  through the formula*

$$\mu = \sum_{\rho} d_{\rho}^2 \quad (18)$$

Instead of presenting more constructed chiral models it may be more informative to mention some of the algebraic methods by which they are constructed and explored. The already mentioned DHR theory provides the conceptual basis for converting the notion of positive energy representation sectors of the chiral model observable algebras  $\mathcal{A}$  (equivalence classes of unitary representations) into localized endomorphisms  $\rho$  of this algebra. This is an important step because contrary to group representations which have a natural tensor product composition structure, representations of operator algebras generally do not come with a natural composition structure. The DHR endomorphisms theory of  $\mathcal{A}$  leads to fusion laws and an intrinsic notion of generalized statistics (for chiral theories: plektonic in addition to bosonic/fermionic). The chiral statistics parameter are complex numbers [11] whose phase is related to a generalized concept of spin via a spin statistics theorem and whose absolute value (the statistics dimension) generalized the notion of multiplicities of fields known from the description of inner symmetries in higher dimensional standard QFTs. The different sectors may be united into one bigger algebra called the exchange algebra  $\mathcal{F}_{red}$  in the chiral context (the “reduced field bundle” of DHR) in which every sector occurs by definition with multiplicity one and the statistics data are encoded into exchange (commutation) relations of charge-carrying operators or generating fields (“exchange algebra fields”) [2]. Even though this algebra is useful in that all properties concerning fusion and statistics are nicely encoded, it lacks some cherished properties of standard field theory namely there is no unique state-field relation i.e. no Reeh-Schlieder property<sup>14</sup>; in operator algebraic terms, the

<sup>14</sup>A field  $A_{\alpha\beta}$  whose source projection  $P_\beta$  does not coalesce with the vacuum projection

local algebras are not factors. This poses the question of how to manufacture from the set of all sectors natural (not necessarily local) extensions with these desired properties. It was found that this problem can be characterized in operator algebraic terms by the existence of so called DHR triples [2]. In case of rational theories the number of such extensions is finite and in the aforementioned “classical” current algebra- and minimal- models they all have been constructed by this method<sup>15</sup>. The same method adapted to the chiral tensor product structure of d=1+1 conformal observables classifies and constructs all 2-dimensional local (bosonic/fermionic) conformal QFT  $\mathcal{B}_2$  which can be associated with the observable chiral input. It turns out that this approach leads to another of those pivotal numerical matrices which encode structural properties of QFT: the coupling matrix  $Z$

$$\mathcal{A} \otimes \mathcal{A} \subset \mathcal{B}_2 \tag{19}$$

$$\sum_{\rho\sigma} Z_{\rho,\sigma} \rho(\mathcal{A}) \otimes \sigma(\mathcal{A}) \subset \mathcal{A} \otimes \mathcal{A}$$

where the second line is an inclusion solely expressed in terms of observable algebras from which the desired (isomorphic) inclusion in the first line follows by a canonical construction, the so-called Jones basic construction. The numerical matrix  $Z$  is an invariant closely related to the so-called *statistics character matrix* [2] and in case of rational models it is even a modular invariant with respect to the modular  $SL(2, Z)$  group transformations (which are closely related to the matrix  $S$  in section 7).

## 6 Integrability, the bootstrap-formfactor program

*Integrability* in QFT and the closely associated *bootstrap-formfactor* construction of a very rich class of massive two-dimensional QFTs can be traced back to two observations made during the 60s and 70s ideas. On the one hand there was the time-honored idea to bypass the “off-shell” field theoretic approach to particle physics in favor of a pure on-shell S-matrix setting which (in particular recommended for strong interactions), as the result of the elimination of short distances via the mass-shell restriction would be free of ultraviolet divergencies. This idea was enriched in the 60s by the crossing property which in turn led to the bootstrap idea, a highly nonlinear seemingly selfconsistent proposal for the determination of the S-matrix. However the protagonists of this S-matrix bootstrap program placed themselves into a totally antagonistic fruitless position with respect to QFT so that the strong return of QFT in the form of gauge theory undermined their credibility. On the other hand there were rather convincing quasiclassical calculations in certain two-dimensional massive QFTs

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annihilates the vacuum.

<sup>15</sup>Thus confirming existing results completing the minimal family by adding some missing models.

as e.g. the Sine-Gordon model which indicated that the obtained quasiclassical mass spectrum is exact and hence suggested that the associated QFTs are integrable (Dashen-Hasslacher-Neveu 1975) and have no real particle creation. These provocative observations<sup>16</sup> asked for a structural explanation beyond quasiclassical approximations, and it became soon clear that the natural setting for obtaining such mass formulas was that of the fusion of boundstate poles of unitary crossing-symmetric purely elastic S-matrices; first in the special context of the Sine-Gordon model (Schroer-Truong-Weisz 1976) and later as a classification program from which factorizing S-matrices can be determined by solving well-defined equations for the elastic 2-particle S-matrix (Karowski-Thun-Truong-Weisz 1977). Some equations in this bootstrap approach resembled mathematical structures which appeared in C. N. Yang's work on non-relativistic  $\delta$ -function particle interactions as well as relations for Boltzmann weights in Baxter's work on solvable lattice models; hence they were referred to as Yang-Baxter relations. These results suggested that the old bootstrap idea, once liberated from its ideological dead freight (in particular from the claim that the bootstrap leads to a unique "theory of everything" (minus gravity)) generates a useful setting for the classification and construction of factorizing two-dimensional relativistic S-matrices. Adapting certain known relations between two-particle formfactors of field operators and the S-matrix to the case at hand (Karowski-Weisz 1978), and extending this with hindsight to generalized (multiparticle) formfactors, one arrived at the axiomatized recipes of the bootstrap-formfactor program of  $d=1+1$  factorizable models (Smirnov 1992). Although this approach can be formulated within the setting of the LSZ scattering formalism, the use of a certain algebraic structure (A.B. and Al. B. Zamolodchikov 1979) which in the simplest version reads

$$\begin{aligned} Z(\theta)Z^*(\theta') &= S^{(2)}(\theta - \theta')Z^*(\theta')Z(\theta) + \delta(\theta - \theta') \\ Z(\theta)Z(\theta') &= S^{(2)}(\theta' - \theta)Z(\theta')Z(\theta) \end{aligned} \quad (20)$$

which will be referred to as the Z-F algebra (Faddeev added the  $\delta$ -term) brought significant simplifications. In the general case the  $Z$ 's are vector-valued and the  $S^{(2)}$ -structure function is matrix-valued<sup>17</sup>. In that case the associativity of the Z-F algebra is equivalent to the Yang-Baxter equations. Recently it became clear that this algebraic relation has a deep physical interpretation; it is the simplest algebraic structure which can be associated with generators of nontrivial wedge-localized operator algebras (see next section).

The mentioned quasiclassical integrability observations also led to another approach which is based on the quantum adaptation of the classical notion of integrability ( $\rightarrow$  (107) Integrable systems: overview). However the construction

<sup>16</sup>It was believed that the "nontrivial elastic scattering implies particle creation" statement of Aks (Aks, 1963) is valid also for low-dimensional QFTs.

<sup>17</sup>The identification of the Z-F structure coefficients with the elastic two-particle S-matrix  $S^{(2)}$  which is prompted by our notation can be shown to follow from the physical interpretation of the Z-F structure in terms of localization..

of a complete (infinite in field theory) set of conserved currents with their associated charges in involution is already a detailed and case-dependent enterprise in the classical setting even before one establishes the absence of quantum anomalies. Conceptually as well as computationally it is much simpler to identify the intrinsic meaning of integrability in QFT with the factorization of its S-matrix or a certain property of wedge-localized algebras (see next section).

The first step of the bootstrap-formfactor program namely the classification and construction of model S-matrices follows a combination of two patterns: prescribing particle multiplets transforming according to group symmetries and/or specifying structural properties of the particle spectrum. The simplest illustration for the latter strategy is supplied by the  $\mathbb{Z}_N$  model. In terms of particle content  $\mathbb{Z}_N$  demands the identification of the  $N^{th}$  bound state with the antiparticle. Since the fusion condition for the bound mass  $m_b^2 = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2m_1m_2ch(\theta_1 - \theta_2)$  is only possible for a pure imaginary rapidity difference  $\theta_{12} = \theta_1 - \theta_2 = i\alpha$  (“binding angle”). Hence the binding of two “elementary” particles<sup>18</sup> of mass  $m$  gives  $m_2 = m \frac{\sin 2\alpha}{\sin \alpha}$  and more generally of  $k$  particles with  $m_k = m \frac{\sin k\alpha}{\sin \alpha}$ , so that the antiparticle mass condition  $m_N = \bar{m} = m$  fixes the binding angle to  $\alpha = \frac{2\pi}{N}$ . The minimal (no additional physical poles) two-particle S-matrix<sup>19</sup> in terms of which the  $n$ -particle S-matrix factorizes is therefore

$$S_{min}^{(2)} = \frac{\sin \frac{1}{2}(\theta + \frac{2\pi i}{N})}{\sin \frac{1}{2}(\theta - \frac{2\pi i}{N})} \quad (21)$$

The  $SU(N)$  models as compared with the  $U(N)$  model requires a similar identification of bound states of  $N-1$  particles with an antiparticle. This S-matrix enters as in the equation for the vacuum to  $n$ -particle meromorphic formfactor of local operators; together with the crossing and the so-called “kinematical pole equation” one obtains a recursive infinite system linking a certain residue with a formfactor involving a lower number of particles. The solutions of this infinite system form a linear space from which the formfactors of specific tensor fields can be selected by a process which is analog but more involved than the specification of a Wick basis of composite free fields. Although the statistics property of two-dimensional massive fields is not intrinsic but a matter of choice, it would be natural to realize e.g. the  $\mathbb{Z}_N$  fields as  $\mathbb{Z}_N$ -anyons.

Another rich class of factorizing models are the Toda theories of which the Sine-Gordon and Sinh-Gordon are the simplest cases. For their descriptions the quasiclassical use of Lagrangians (supported by integrability) turns out to be of some help in setting up their more involved bootstrap-formfactor construction.

The unexpected appearance of objects with new fundamental (solitonic) charges (example: the Thirring field as the carrier of a solitonic Sine-Gordon charge) or the unexpected confinement of charges (example: the  $CP(1)$  model

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<sup>18</sup>The quotation mark is meant to indicate that in contrast to the Schroedinger QM there is “nuclear democracy” on the level of particles. The inexorable presence of interaction-caused vacuum polarization limits a fundamental/fused hierarchy to the fusion of charges.

<sup>19</sup>minimal=without so-called CDD poles

as a confined  $SU(2)$  model) turns out to be opposite sides of the same coin and both cases have realizations in the setting of factorizing models [2].

## 7 Factorizing QFT, PFGs and lightray holography

There are two recent ideas which place the two-dimensional bootstrap-formfactor program into a more general setting which permits to understand its position in the general context of local quantum physics.

Let us restrict our interest to models which fulfill the standard assumption of LSZ scattering theory (mass gap, asymptotic completeness) and assume for simplicity just kind of particle. Let  $G$  be a (generally unbounded) operator affiliated with the local algebra  $\mathcal{A}(\mathcal{O})$ . We call such  $G$  a vacuum-polarization-free generator (PFG) affiliated with  $\mathcal{A}(\mathcal{O})$  (denoted as  $G\eta\mathcal{A}(\mathcal{O})$ ) if the state vector  $G\Omega$  (with  $\Omega$  the vacuum) is a one-particle state without any vacuum polarization admixture [2]. PFGs are by definition (unbounded) *on-shell* operators and it is well-known that the existence of a *subwedge-localized* PFG forces the theory to be free, i.e. the local algebras in such a situation are generated by free fields. However, and this is the surprising fact, PFGs and interactions are compatible in wedge regions. Such localization regions offer the best compromise between particles localization and vacuum-polarization favoring field localization<sup>20</sup>. Although modular operator theory guaranties the existence of wedge generators without vacuum polarization, these PFG have useful properties in the setting of time-dependent scattering theory only if they are “tempered” (well-defined on a translation invariant domain) [2]. The restriction implied by this additional requirement can be shown to only permit theories with a purely elastic S-matrix and it has been known for a long time that this is possible only in  $d=1+1$  where such theories have been investigated since the late 70s within the bootstrap-formfactor program. In fact on the basis of formfactor properties one can show that the elastic two-dimensional S-matrices coming from a local QFT are already described by a two-particle S-matrices [2], all the higher elastic contributions factorize into two-particle contributions and the latter are classified by solving equations (parametrized in terms of rapidities) which incorporate unitarity, analyticity and crossing [2]. The second surprise is that the Fourier transforms of the wedge-algebra-generating tempered PFGs are identical to operators introduced at the end of the 70s by Zamolodchikov (their properties were spelled out in more detail by Faddeev). Although their usefulness in the bootstrap formfactor program was beyond doubt, their conceptual position within QFT was not clear since, notwithstanding their formal similarity to free field creation/annihilation operators, their physical content is distinctively different from incoming or outgoing free fields of scattering theory. In the simple case (20) of just one interacting particle (without boundstates e.g.

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<sup>20</sup>In any smaller localization region the interaction-caused vacuum polarizations would only permit field- but not particle- localization.

the “sinh-Gordon” model) the generators of the wedge algebra are of the form

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int (e^{ip(\theta)x(\chi)} Z(\theta) + h.c.) d\theta \quad (22)$$

This defines a covariant field which, although not being pointlike local maintains some localization; it turns out to be wedge-like local<sup>21</sup> i.e. it commutes with its “modular opposite”  $J\phi(x)J$  which generates the causal disjoint algebra  $\mathcal{A}(W') = \mathcal{A}(W)'$ . This interpretation of the Z-F algebra operators in terms of localization concepts turns out to be a valuable starting point for the construction of tighter localized algebras  $\mathcal{A}(D)$  associated with double cone regions  $D$  by computing intersections of wedge algebras whose generating operators turn out to be infinite series in the  $Z$ 's with coefficient functions which are generalized formfactors. The difficult problem of demonstrating the existence of a QFT associated with the algebraic structure (20) of wedge algebra generators (22) is then encoded into a nontriviality statement for the double cone intersections ( $\mathcal{A}(D) = \mathcal{A}(W_a) \cap \mathcal{A}(W)' \neq C\mathbf{1}, W_a$  translated wedge); the nontriviality of these intersections for arbitrary small  $D$  corresponds to the existence of pointlike generating fields in the setting of Wightman. The important difference to the standard perturbative Lagrangian quantization approach is that the computations of intersections do not require the use of singular correlation functions of pointlike distributional objects; fields are simply size independent pointlike generators of local algebras which are convenient for coordinatizing the resulting local nets of operator algebras but should be avoided in actual computations where they create the ultraviolet divergence menace. Unfortunately the standard quantization approach is not able to do this. The modular wedge localization formalism in terms of PFG generators fulfilling the Z-F algebra relation for the first time permits to bypass quantization and to walk without “classical crutches” in case of a special interacting class of factorizable QFT which in their renormalized perturbative quantization treatment would present all the ultraviolet problems characteristic of the standard approach. In the absence of interactions Wigner already achieved an intrinsic formulation in his 1939 one-particle theory. In fact there are good reasons for viewing the present ideas as an extension of Wigner’s approach into the realm of interactions which as the result of interaction-induced vacuum polarization delocalize particles and make the introduction of more convenient carriers of localization unavoidable.

The recognition that the knowledge of the position of a wedge-localized subalgebra  $\mathcal{A}(W)$  with  $\mathcal{A}(W)' = \mathcal{A}(W')$  within the full Fock space algebra  $B(H)$  and the action of the representation of the Poincaré group in  $B(H)$  on the  $\mathcal{A}(W)$  determines the full net of algebras  $\mathcal{A}(\mathcal{O})$  via intersections

$$\mathcal{A}(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W) \quad (23)$$

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<sup>21</sup>The  $x$  continues to comply with the covariant transformation law but it is not a point of localization i.e. the smearing with wedge supported test functions  $\phi(f)$  does not lead to an improvement in localization if one reduces the support of  $f$ .

is actually independent of spacetime dimensions and factorizability. But only in  $d=1+1$  within the setting of factorizable models one finds simple PFG generators for  $\mathcal{A}(W)$  which permit the computation of intersections.

With the particle picture outside factorizing theories being made less useful by de-localization through interaction induces vacuum polarization [2] it is encouraging to notice that there is another constructive idea based on modular inclusion and intersections which does not require the very restrictive presence of wedge-localized PFGs.. This is the holographic projection to the lightfront. It maps a massive (non-conformal) QFT to a (transverse) extended chiral theory on the lightfront  $x_- = 0$  in such a way that the global original algebra on Minkowski spacetime  $\mathcal{A}(M) = B(H)$  and its global holographic lightfront projection  $\mathcal{A}(LF) = B(H)$  coalesce<sup>22</sup>, but the local substructure (the spacetime-indexed net) is radically different, apart from wedge-localized algebras which are identical to the algebras of their upper lightfront boundary of  $W$ ,  $\mathcal{A}(W) = \mathcal{A}(LF(W))$ . The situation simplifies considerably in massive two-dimensional theories in which case the transverse extension is absent and the holographic lightray projection leads to bona-fide chiral theories (i.e. Moebius-invariant theories whose extendibility to  $Diff(S^1)$  will be the subject of some remarks in the next section). The restriction to two-dimensional factorizing theories leads to further significant simplifications; within this class the holographic projection to chiral theories has a unique inversion. If the  $Diff(S^1)$  transformations are present in the holographic projection, they were already present in the ambient theory. The reason why they are not noticed in the spacetime indexing of the ambient net is that they acts in a nonclassical “fuzzy” manner and hence escapes the standard symmetry formalism via Noether currents and their quantization [2]. The holographic relation has very interesting consequences for those chiral models (all?) whose observable algebras are in the holographic image of factorizing massive theories because they can be characterized in terms of a Z-F algebra structure which in view of its simple systematics (and the fact that there exists no direct Lie-algebraic structural characterization for general chiral models) may turn out to provide valuable additional help in their classification.

Besides the rigorous one-to-one relation of factorizing theories to their chiral holographic projections there is of course also the critical- or scale-invariant limit which leads to an associated two-dimensional conformal invariant theory. The idea that the critical universality classes are much smaller within the setting of factorizing models is the starting point of a Zamolodchikov’s successful proposal to approach the classification and identification of factorizing models via perturbing the better known conformal models by selecting particular perturbations in terms of chiral operators. Of course even in the limited factorized setting one cannot expect a one-to one correspondence since the relation to the conformal massless limits is not just a simple mass-dressing of fields which already exist on the massless level (examples show that there are fields in the massive model which vanish if the massless limit is taken as in section3). However for

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<sup>22</sup>Conformally invariant theories in  $d=1+1$  are the only exception to this equality; as a result of the chiral factorization one need the characteristic data on both lightrays.



the consistency of the Zamolodchikov perturbation scenario one does not have to construct *all* massive fields directly; it suffices that the ones which disappear in the massless limit are composites of those who persist. The universality class division of massive theories by the scale-invariant limit is conceptually very different from that of the holographic projection; the latter involves a different encoding of spacetime indexing, but does not affect the algebraic “stem cell substrate”<sup>23</sup> which can be grown into different QFTs by changing the (curved) space-time indexing (see also next section).

It turns out that the plektonic relations of charged fields and the issue of statistics of particles loose their physical relevance for two-dimensional massive models since they can be changed without affecting the physical content. Instead such notions as order/disorder fields and soliton take their place [2].

In accordance with its historical origin the theory of two-dimensional, factorizing models may also be is an outgrowth of the quantization of classical integrable systems ( $\rightarrow$ (363) *Integrable models in two dimensions*). But in comparison with the rather involved structure of integrability (existence of sufficiently many commuting conservation laws), the conceptual setting of factorizing models within the scattering framework (factorization $\simeq$ existence of wedge-localized tempered PFGs) is rather simple and intrinsic.

There are many additional important observations on factorizing models whose relation to the physical principles of QFT, unlike the bootstrap-formfactor program, is not yet settled. The meaning of the  $c$ -parameter outside the chiral setting and ideas on its renormalization group flow as well as the various formulations of the thermodynamic Bethe Ansatz belong to a series of interesting observations whose final relation to the principles of QFT still needs clarification.

## 8 Ongoing research, results from operator algebra methods

QFT has been enriched by a the powerful new concept of modular localization which promises to revolutionize the task of (nonperturbative) classification and construction of models. It provides an additional strong link between two-dimensional and higher dimensional QFT and admits a rich illustration for chiral and factorizing theories. In the following we comment on two such ongoing investigations.

One is motivated by the recent discovery of the adaptation of Einsteins classical principle of local covariance to QFT in curved spacetime. The central question raised by this work ( $\rightarrow$  (78) Algebraic approach to quantum field theory) is if all models of Minkowski spacetime QFTs permit a local covariant extension to curved spacetime and if not which models do. In the realm of chiral QFT this would amount to ask if all Moebius-invariant models are also

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<sup>23</sup>I find this analogy quite helpful for a more intrinsic understanding of how QFT processes the abstract algebraic substrate into various different spacetime-indexed algebraic nets.

$\text{Diff}(S^1)$ -covariant.

The second one concerns the operator algebraic interpretation of temperature duality which includes the Verlinde relation as a special case. This requires the elaboration of a chiral analog of the Osterwalder-Schrader Euclideanization.

## 8.1 Spacetime symmetries from the relative positions of monades

Localized operator algebras  $\mathcal{A}(\mathcal{O})$  for spacetime region  $\mathcal{O}$  with a nontrivial causal disjoint  $\mathcal{O}'$  are under very general conditions (for wedge-localized algebras and interval localized algebras of chiral QFT no additional conditions need to be imposed) isomorphic to a unique algebra whose special role was highlighted in mathematical work by Connes and Haagerup and whose physical *raison d'être* is the inexorable vacuum polarization associated with relativistic localization. It is quite surprising that the full richness of QFT can be encoded into the relative position of a finite number of copies of this “monade”<sup>24</sup> within a common Hilbert space [2]. Chiral conformal field theory offers the simplest theoretical laboratory in which this issue can be analysed.

If the modular group  $\sigma_t^{\mathcal{B}}$  in a *joint standard* situation for an inclusion  $(\mathcal{A} \subset \mathcal{B}, \Omega)$  of two monades (which share one  $\Omega$ ) acts on the smaller algebra for  $t < 0$  as a one-sided compression  $\sigma_t^{\mathcal{B}}(\mathcal{A}) \subset \mathcal{A}$ , the two modular unitaries  $\Delta_{\mathcal{A}, \mathcal{B}}^{it}$  generate a unitary representation of a positive energy spacetime translation-dilation (Anosov) group with the commutation relation (Borchers, Wiesbrock 1992/93)

$$\text{Dil}(\lambda)U(a)\text{Dil}^*(\lambda) = U(\lambda a), \quad \text{Dil}(e^{-2\pi t}) = \Delta_{\mathcal{B}}^{it} \quad (24)$$

The geometrical picture which goes with this abstract modular inclusion is  $\mathcal{B} = \mathcal{A}(I) \supset \mathcal{A}(\tilde{I}) = \mathcal{A}$  with the two intervals  $\tilde{I} \subset I$  having one endpoint in common so that the modular group of the bigger one  $\simeq \text{Dil}_I$  (Möbius transformation leaving  $\partial I$  fixed) leaves this endpoint invariant and compresses  $\tilde{I}$  into itself by transforming the other endpoint of  $\partial \tilde{I}$  into  $I'$ . One can show that this half-sided modular inclusion situation ( $\pm hsm$ ,  $t \lesseqgtr 0$ ) actually cannot result from any other von Neumann type than copies of the monade.

The simplest way to obtain the full Möbius group as a symmetry group of a vacuum representation from pure operator-algebraic data is to require that the modular inclusion itself is standard which means that in addition the vacuum  $\Omega$  is also standard with respect to the relative commutant  $\mathcal{A}' \cap \mathcal{B}$  (the third monade). The associated geometric picture is that of two half-circles whose intersection is a quarter circle [2].

**Theorem 2** *The observable algebras of chiral QFT are classified by standard hsm of two monades.*

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<sup>24</sup>We borrow this terminology from the mathematician and philosopher Gottfried Wilhelm Leibniz; in addition to its intended philosophical content (reality created by relations between monades) it has the advantage of being much shorter than the mathematical terminology “hyperfinite type III<sub>1</sub> Murray-von Neumann factor”. Instead of “a finite number of copies of the (abstract) monade”, we will simply say “a finite number of monades”

The net of interval-indexed local observable algebras is obtained by applying the Moebius group to the original monade  $\mathcal{A}$  or  $\mathcal{B}$  and the problem of classifying chiral models is reduced to a well-defined problem in the theory of operator algebras.

Encouraged by this successful encoding of the vacuum symmetry group of chiral theories into the relative position of monades, it is natural to ask whether this algebraic encoding can be extended to the vacuum-changing part of  $Diff(S)$ , which is what the principle of local covariance would require. Certainly all of the afore-mentioned models permit this extension since they possess an stress-energy tensor whose Fourier decomposition leads to the unitary implementation of  $Diff(S)$ . The known counterexamples [2] of models which are Moebius invariant but lack the full  $Diff(S^1)$  covariance can be excluded on the basis of two well-motivated quantum physical properties: *strong additivity* and the *split property* [2]. So it is natural to ask whether these local quantum physical requirements guaranty the extension  $Moeb \rightarrow Diff(S^1)$  from the global vacuum preserving Moebius invariance to local  $Diff(S)$  covariance. This is indeed possible if and only if in addition to the vacuum there exist other state vectors  $\Phi$  which with respect to certain (multi-) local subalgebras  $\mathcal{A}(I)$  lead to standard pairs  $(\mathcal{A}(I), \Phi)$  whose modular group is *partially geometric*. For a presentation of this concept and its role in the extension problem see [2].

**Theorem 3** *For strongly additive Moebius-invariant chiral models which fulfill the split property, the  $Diff_2(S)$  covariance is equivalent to the existence of a partially geometric non-vacuum vectors  $\Psi$  such that the modular group of  $(\mathcal{A}(I) \vee \mathcal{A}(J), \Psi)$  acts as  $Dil_2$  on  $I \cup J$ .*

The 3-parametric group  $Diff_2(S)$  results from  $Dil_2$  by changing the position of the fixpoints through the application of Moebius transformations and defining the group generated by these generalized dilations. The analogous use of the  $k^{th}$  instead of the  $2^{nd}$  power leads to  $Dil_k$  restricted to a  $k$ -fold localized algebra; with  $k$ -fold localized intervals placed into a more general positions one generates  $Diff_k(S)$ . With this construction we have reached our aim to encode the geometric extension problem to  $Diff(S)$  into a local quantum physical requirement. Whether these required modular properties for securing the existence of the non-vacuum preserving part can also be encoded into an algebraic significant positioning of monades is presently not known.

The problem of characterizing Poincaré (or conformal) invariant higher dimensional QFTs in terms of a finite number of monades has a positive answer; in this case the local covariance principle has however only been only checked for the Weyl algebra as well as in the perturbative approach to QFT on curved spacetime [2]. It is hard to imagine how one can combine quantum theory and gravity without understanding first the still mysterious links between spacetime geometry, thermal properties and relative positioning of monades in a joint Hilbert space.

## 8.2 Euclidean rotational chiral theory and temperature duality

Euclidean theory associated with certain real time QFTs is a subject whose subtle and restrictive nature has largely been lost in many contemporary publications as a result of the “banalization” of the Wick rotation (for some pertinent critical remarks see [2]). The mere presence of analyticity linking real with imaginary (Euclidean) time, without establishing the subtle reflection positivity (which is necessary to derive the real time spacelike commutativity as well as the Hilbert space structure), is not of much physical use; what is needed is an operator algebraic understanding of the so-called Wick rotation.

The issue of understanding Euclideanization in chiral theories became particularly pressing after it was realized that Verlinde’ observation on a deep structural connection between fusion rules and modular transformation properties of characters of irreducible representations of chiral observable algebras is best taken care of by considering it as a part of a wider setting involving angular parametrized thermal n-point correlation functions in the superselection sector  $\rho_\alpha$

$$\langle A(\varphi_1, \dots, \varphi_n) \rangle_{\rho_\alpha, 2\pi\beta_t} := \text{tr}_{H_{\rho_\alpha}} e^{-2\pi\beta_t(L_0^{\rho_\alpha} - \frac{c}{24})} \pi_{\rho_\alpha}(A(\varphi_1, \dots, \varphi_n)) \quad (25)$$

$$A(\varphi_1, \dots, \varphi_n) = \prod_{i=1}^n A_i(\varphi_i)$$

i.e. the Gibbs trace at inverse temperature  $\beta = 2\pi\beta_t$  on observable fields in the representation  $\pi_{\rho_\alpha}$ . Such thermal states are (in contrast to the previously used ground states) independent on the particular localization  $\text{loc}\rho_\alpha$ , they only depend on the equivalence class i.e. on the sector  $[\rho_\alpha] \equiv \alpha$ . These correlation functions<sup>25</sup> fulfill the following thermal duality relation

$$\langle A(\varphi_1, \dots, \varphi_n) \rangle_{\alpha, 2\pi\beta_t} = \left(\frac{i}{\beta_t}\right)^a \sum_{\gamma} S_{\alpha\gamma} \left\langle A\left(\frac{i}{\beta_t}\varphi_1, \dots, \frac{i}{\beta_t}\varphi_n\right) \right\rangle_{\gamma, \frac{2\pi}{\beta_t}} \quad (26)$$

where the right hand side formally is a sum over thermal expectation at the inverse temperature  $\frac{2\pi}{\beta_t}$  at the analytically continued pure imaginary values scaled with the factor  $\frac{1}{\beta_t}$ . The multiplicative scaling factor in front which depends on the scaling dimensions of the fields and is just the one which one would naively write if the transformation  $\varphi \rightarrow \frac{i}{\beta_t}\varphi$  were an admissible conformal transformation law.

This relation can be checked explicitly (using Poisson resummation techniques) in simple models as the abelian current models [2]. Since the Gibbs states are not normalized, the Kac-Peterson-Verlinde character identities are actually the “zero-point function” part (i.e.  $A = 1$ ) of the above relation (with the statistics character matrix  $S$  already mentioned at the end of section 4). A

<sup>25</sup>The conformal invariance actually allows a generalization to complex Gibbs parameters  $\tau$  with  $\text{Im}\tau = \beta$  which is however not needed in the context of the present discussion.

model-independent derivation of (26) can be given in the operator algebraic setting of *angular Euclideanization*. This theory leads to a map which takes a dense analytic subalgebra of  $\mathcal{A}(S^1)$  into one of a “Euclidean” theory which apart from having a different Hilbert space inner product and consequently a different star operation but for which the respective closures are analogous. In the ensuing identity between the correlation functions of pointlike covariant generators (26) the statistics character matrix  $S$  enters in an interesting way and together with another diagonal phase matrix  $T$  leads to a situation in which the discrete modular group  $SL(2, R)$  plays the role of a new  $SL(2, R)$  symmetry-like structure [2].

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